

Bi-Hamiltonian Structure in Serret-Frenet Frame

E Abadoğlu and H. Gümral
 Department of Mathematics, Yeditepe University
 Kayışdağı 34750 İstanbul Turkey

eabadoglu@yeditepe.edu.tr, hgumral@yeditepe.edu.tr
 Nov 15, 2007

Abstract We reduced the problem of constructing bi-Hamiltonian structure in three dimensions to the solution of a Riccati equation in moving coordinates of Serret-Frenet frame. We then show that either the linearly independent solutions of the corresponding second order equation or the normal vectors of the moving frame imply two compatible Poisson structures.

1 Introduction

The discovery of completely integrable nonlinear evolution equations as well as the algebraic and geometric structures associated with them has triggered an intensive search of finite dimensional dynamical systems resembling the similar properties. The bi-Hamiltonian structure as an underlying geometrical framework for complete integrability provoked the revival of Poisson structures of finite dimensional dynamical systems (see [1] and the references therein for details and comparison).

Several works [2]-[9] on construction of conserved quantities, on Hamiltonian structures and on integrability of three dimensional systems have led to a systematic investigation using Poisson geometry, Frobenius integrability theorem and unfoldings of foliations [10], [11]. We presented correspondence between Poisson structures and integrable one forms and utilized this to obtain criteria for local and global existence of Poisson structures. We also obtained the local result that any two Poisson structures can be made into a compatible pair to form a bi-Hamiltonian structure.

The restrictions on Poisson matrices imposed by the Jacobi identity and the compatibility condition for two of them to form a bi-Hamiltonian structure are the most serious conditions requiring deliberate actions against their simple presence as one scalar equation in three dimensions [12]-[15]. In [11], using an invariance property, we reduced the Jacobi identity to a nonlinear equation in ratios of components of the Poisson matrix. This scalar equation in one unknown function was shown to contain sufficient information for constructing the Poisson structure completely and was recognized to be the Riccati equation in [16].

The possibility to determine the Poisson matrix by a single function straightens out the difficulties in general Hamiltonian systems which does not fall into

the classes of canonical Hamiltonian or Lie-Poisson (i.e. with linear Poisson structure) equations, arising from the absence of coordinates similar to the canonical Darboux coordinates of symplectic geometry. The Darboux-Weinstein theorem [17] describes the local structure of a Poisson manifold as a space foliated by symplectic submanifolds. The foliation depends on the rank of the structure which is an invariant of the Poisson matrix. As a result, the Poisson matrix consists of a constant submatrix whose rank is the rank of the Poisson structure and some additional nonlinear part. In three dimensions, one must solve for at least one unknown function to determine the Poisson matrix completely.

The role of the constants of motion, in particular the Casimirs, in linearization and integration of the equations associated with the Jacobi identity were also discussed in [12]-[16]. Yet, there is no direct relations of the Jacobi identity and the compatibility condition to the theory of linear differential equations which may be one of the elegant ways to avoid some deceptive conclusions from these simple looking differential equations ignoring their nonlinear character. The main source of these confusions is to endeavor to exploit local criteria for global results without questioning any suspicious obstructions such as the one we have found for the Darboux-Halphen system [11].

In [11] we also observed through several examples that some coordinate transformations may cast the dynamical systems into a form where, in spite of much higher degree of nonlinearity, the integration for the conserved quantities and hence the manifestation of the bi-Hamiltonian structures become more efficient. The non-covariance of the general Hamiltonian formulation is a source of the common belief that the existence of Hamiltonian structure and the integrability of dynamical systems rely heavily upon the coordinates in which they are represented as well as the preferred parametrization of the solution curves. Such coordinates were found to be important in numerical integrations as well. It is shown in [18] that, a necessary condition for numerical solution algorithms to preserve the conserved quantities of dynamical systems is their resemblance, as products of a skew symmetric matrix and a gradient vector, to Hamiltonian systems. See also [19] for numerical schemes applied to some concrete examples of systems under consideration. In three dimensions, the Nambu mechanics [2] is the only and generic (up to a conformal factor) framework which enables us to identify such coordinates. Namely, the Nambu structure is a manifestation of the bi-Hamiltonian structure, hence integrability, in a frame with coordinate vectors consisting of the dynamical vector field and gradients of two conserved Hamiltonians [11].

The Nambu representation obviously requires the integration of the system for Hamiltonian functions. In our study of the Darboux-Halphen system, we were able to obtain obstruction for the global integration of such quantities [11]. To our knowledge, this is the only example which exhibits, along with a rich geometric structure, differentiation between local criteria and global availability. On the other hand, the local version of the Nambu mechanics has not yet been appeared in the literature.

In this work, we shall show that the coordinates associated with the Serret-

Frenet frame is the one we sought for three dimensional systems. In these coordinates the Jacobi identity linearizes through the Riccati equation and the compatibility follows from the Hamilton's equations. Obstructions to the global constructions of the bi-Hamiltonian structures are encoded in the helicities and the cross-helicity of the unit vectors spanning the normal plane to the vector field associated with the dynamical system. We shall construct a bi-Hamiltonian moving frame for the local Nambu representation.

In the next section we review the properties of bi-Hamiltonian systems in three dimensions. In section three we introduce the Serret-Frenet frame associated with a vector in three dimension. We express the Jacobi identity in Serret-Frenet frame and show that in moving coordinates it reduces to a Riccati equation. In section four, we shall show that Poisson structures constructed from the solutions of the Riccati equations and/or the normal vectors are all compatible via Hamilton's equations of motion. We then conclude the existence of bi-Hamiltonian structures.

2 Hamiltonian Systems in Three Dimensions

Following [11], we shall summarize the necessary ingredients of the bi-Hamiltonian formalism in three dimensions. For $\mathbf{x} = \{x^i\} = (x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$ and dot denoting the derivative with respect to t , we consider the autonomous differential equations

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}) \quad (1)$$

associated with a three-dimensional smooth vector field \mathbf{v} . This equation is said to be Hamiltonian if the vector field can be written as

$$\mathbf{v}(\mathbf{x}) = \Omega(\mathbf{x})(dH(\mathbf{x})) \quad (2)$$

where $H(\mathbf{x})$ is the Hamiltonian function and $\Omega(\mathbf{x})$ is the Poisson bi-vector (skew-symmetric, contravariant two-tensor) subjected to the Jacobi identity

$$[\Omega(\mathbf{x}), \Omega(\mathbf{x})] = 0 \quad (3)$$

defined by the Schouten bracket. In coordinates, if $\partial_i = \partial/\partial x^i$ the Poisson bi-vector is $\Omega(\mathbf{x}) = \Omega^{jk}(\mathbf{x}) \partial_j \wedge \partial_k$ with summation over repeated indices. Then the Jacobi identity reads

$$\Omega^{i[j} \partial_i \Omega^{kl]} = 0 \quad (4)$$

where $[jkl]$ denotes the antisymmetrization over three indices. It follows that in three dimensions the Jacobi identity is a single scalar equation. One can exploit the vector calculus and the differential forms in three dimensions to have a more transparent understanding of Hamilton's equations as well as the Jacobi identity. Using the isomorphism

$$J_i = \varepsilon_{ijk} \Omega^{jk} \quad i, j, k = 1, 2, 3 \quad (5)$$

between skew-symmetric matrices and (pseudo)-vectors defined by the completely antisymmetric Levi-Civita tensor ε_{ijk} we can write the Hamilton's equations (2) in vector form

$$\mathbf{v} = \mathbf{J} \times \nabla H \quad (6)$$

and in this notation the Jacobi identity (5) becomes

$$\mathbf{J} \cdot (\nabla \times \mathbf{J}) = 0 \quad (7)$$

In this form, the Jacobi identity is recognized to be equivalent to the Frobenius integrability condition for the vector \mathbf{J} , or equivalently, the condition for the one form $J = J_i dx^i$ to define a foliation of codimension one in three dimensional space [20], [21], [11].

A distinguished property of Poisson structures in three dimensions is the invariance of the Jacobi identity under the multiplication of the Poisson vector $\mathbf{J}(\mathbf{x})$ by an arbitrary but non-zero factor. More precisely, one can easily show that under the transformation

$$\mathbf{J}(\mathbf{x}) \rightarrow f(\mathbf{x})\mathbf{J}(\mathbf{x}) \quad (8)$$

of Poisson vector the Jacobi identity transforms as

$$\mathbf{J} \cdot (\nabla \times \mathbf{J}) \rightarrow (f(\mathbf{x}))^2 \mathbf{J} \cdot (\nabla \times \mathbf{J}) \quad (9)$$

which manifests the invariance property. The identities

$$\mathbf{J} \cdot \mathbf{v} = 0, \quad \nabla H \cdot \mathbf{v} = 0 \quad (10)$$

follows directly from the Hamilton's equations (6), the second of which is the expression for the conservation of the Hamiltonian function.

A three dimensional vector $\mathbf{v}(\mathbf{x})$ is said to be bi-Hamiltonian if there exist two different compatible Hamiltonian structures. In the notation of equation (6), this implies

$$\mathbf{v} = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1 \quad (11)$$

for the dynamical equations. The compatibility condition for \mathbf{J}_1 and \mathbf{J}_2 is defined by the Jacobi identity for the Poisson vector $\mathbf{J}_1 + c\mathbf{J}_2$ for arbitrary constant c . Namely, \mathbf{J}_1 and \mathbf{J}_2 are compatible Poisson vectors provided they satisfy

$$\mathbf{J}_1 \cdot (\nabla \times \mathbf{J}_2) + \mathbf{J}_2 \cdot (\nabla \times \mathbf{J}_1) = 0. \quad (12)$$

The invariance properties of the Jacobi identity and the Hamiltonian functions enable one to extend the constant c to be a function of the conserved Hamiltonians. More precisely, the Jacobi identity for the combination $\mathbf{J}_1 + c\mathbf{J}_2$ of Poisson vectors gives

$$(\mathbf{J}_1 \times \mathbf{J}_2) \cdot \nabla c = (\mathbf{J}_1 \cdot (\nabla \times \mathbf{J}_2) + \mathbf{J}_2 \cdot (\nabla \times \mathbf{J}_1))c \quad (13)$$

which reduces to equation (12) whenever c is a constant. This linear equation can always be solvable for the function c resulting in considerable relaxation in

the compatibility condition. That means, locally every pair of Poisson vectors can be made compatible.

It follows from the bi-Hamiltonian equations (11) and the identities (10) that $\mathbf{J}_1 \times \nabla H_1 = \mathbf{J}_2 \times \nabla H_2 = 0$. That is, the Hamiltonian of one structure is the Casimir function of the other. Thus, a three dimensional dynamical system can be defined to be integrable if it is a Hamiltonian system with one Casimir. In this case, the flow can be represented by the intersection of surfaces defined by constant values of the integrals of motion (see [11] and references below for further details and examples).

3 Jacobi Identity in Serret-Frenet Frame

Let $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ denote the Serret-Frenet frame associated with a differentiable curve $t \rightarrow \mathbf{x}(t)$ in some domain of the three dimensional space \mathbb{R}^3 . Throughout, $\nabla = (\partial_x, \partial_y, \partial_z)$ will denote the usual gradient operator in local Cartesian coordinates. Given a vector field \mathbf{v} , the unit tangent vector \mathbf{t} , the unit normal \mathbf{n} , and the unit bi-normal \mathbf{b} can be constructed immediately as

$$\mathbf{t}(\mathbf{x}) = \frac{\mathbf{v}(\mathbf{x})}{\|\mathbf{v}(\mathbf{x})\|} \quad \mathbf{n}(\mathbf{x}) = \frac{\mathbf{t} \times (\nabla \times \mathbf{t})}{\|\mathbf{t} \times (\nabla \times \mathbf{t})\|} \quad \mathbf{b}(\mathbf{x}) = \mathbf{t}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) \quad (14)$$

and they form a right-handed orthonormal frame except those vector fields \mathbf{v} satisfying the condition imposed by

$$\mathbf{t} \times (\nabla \times \mathbf{t}) = 0. \quad (15)$$

It can be deduced from the vector identity $2\mathbf{t} \times (\nabla \times \mathbf{t}) = \nabla(\mathbf{t} \cdot \mathbf{t}) + 2\mathbf{t} \cdot \nabla \mathbf{t} = 2\mathbf{t} \cdot \nabla \mathbf{t}$ that this condition excludes essentially the flows with constant unit tangent and the points \mathbf{x} at which the unit normal \mathbf{n} (hence the bi-normal \mathbf{b}) have zeros. That is, the cases one cannot have a Serret-Frenet frame. To avoid this we may assume that

$$(\nabla \times \mathbf{t}) \neq \lambda(\mathbf{x}) \mathbf{t} \quad (16)$$

for arbitrary nonzero function $\lambda(\mathbf{x})$. That is, we exclude the dynamical systems whose unit tangent vectors are the eigenvectors of the curl operator [22], [23].

We introduce the directional derivatives along the triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ as

$$\partial_s = \mathbf{t} \cdot \nabla \quad \partial_n = \mathbf{n} \cdot \nabla \quad \partial_b = \mathbf{b} \cdot \nabla \quad (17)$$

so that the variables (s, n, b) are the coordinates associated with the Serret-Frenet frame. By inverting equations (17) we get the expression

$$\nabla = \mathbf{t} \partial_s + \mathbf{n} \partial_n + \mathbf{b} \partial_b \quad (18)$$

for the Cartesian gradient in Serret-Frenet frame. Since $\mathbf{t} \times \nabla \times \mathbf{t} = \mathbf{t} \cdot \nabla \mathbf{t} = \partial_s \mathbf{t}$ the definition of the normal vector reduces to one of the Serret-Frenet equations justifying the name for the moving frame introduced [24].

It follows from the identity in equation (10) that the Poisson vector \mathbf{J} has no component along the unit tangent vector \mathbf{t} . Hence, we set

$$\mathbf{J} = \alpha \mathbf{n} + \beta \mathbf{b} \quad (19)$$

for unknown functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ satisfying $\alpha^2 + \beta^2 \neq 0$. Using derivatives in Cartesian variables we find the expression

$$\begin{aligned} \mathbf{J} \cdot (\nabla \times \mathbf{J}) = & (\beta \nabla \alpha - \alpha \nabla \beta) \cdot \mathbf{t} + \alpha^2 \mathbf{n} \cdot (\nabla \times \mathbf{n}) + \beta^2 \mathbf{b} \cdot (\nabla \times \mathbf{b}) \\ & + \alpha \beta (\mathbf{n} \cdot (\nabla \times \mathbf{b}) + \mathbf{b} \cdot (\nabla \times \mathbf{n})) \end{aligned} \quad (20)$$

for the Jacobi identity. Assuming $\alpha \neq 0$ and defining the function $\mu = \beta/\alpha$ the Jacobi identity for $\alpha(\mathbf{n} + \mu \mathbf{b})$ gives

$$\mathbf{t} \cdot \nabla \mu = \mathbf{n} \cdot \nabla \times \mathbf{n} + \mu (\mathbf{n} \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{n}) + \mu^2 \mathbf{b} \cdot \nabla \times \mathbf{b} \quad (21)$$

which is an equation involving only the unknown function μ . Obviously, this simplification is a manifestation of the invariance of the Jacobi identity under the multiplication of \mathbf{J} by an arbitrary but non-zero function. Similarly, we may assume $\beta \neq 0$ and define $\eta = -1/\mu = -\alpha/\beta$ for which the Jacobi identity for the combination $\beta(\mathbf{b} - \eta \mathbf{n})$ becomes

$$\mathbf{t} \cdot \nabla \eta = \mathbf{b} \cdot \nabla \times \mathbf{b} - \eta (\mathbf{n} \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{n}) + \eta^2 \mathbf{n} \cdot \nabla \times \mathbf{n}. \quad (22)$$

To this end, we define the scalar quantities measuring the non-integrability (in the sense of Frobenius) of each unit vector in the Serret-Frenet triad

$$\Omega_{\mathbf{t}} = \mathbf{t} \cdot (\nabla \times \mathbf{t}), \quad \Omega_{\mathbf{n}} = \mathbf{n} \cdot (\nabla \times \mathbf{n}), \quad \Omega_{\mathbf{b}} = \mathbf{b} \cdot (\nabla \times \mathbf{b}) \quad (23)$$

the first of which is necessarily not equal to any eigenvalue of the curl operator by the assumption 16. The integration of these quantities over the three space gives the so called helicities [25], [26] associated with the triad. We also introduce the sum

$$\Omega_{\mathbf{nb}} = \mathbf{n} \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{n} \quad (24)$$

of the cross-helicities for the normal and bi-normal vectors.

Proposition 1 *Let $\mathbf{J} = \alpha(\mathbf{n} + \mu \mathbf{b})$ (or $\mathbf{J} = \beta(\mathbf{b} - \eta \mathbf{n})$). Then, the Jacobi identity for \mathbf{J} in moving coordinates is given by the Riccati equation*

$$\partial_s \mu = \Omega_{\mathbf{n}} + \mu \Omega_{\mathbf{nb}} + \mu^2 \Omega_{\mathbf{b}} \quad (25)$$

$$(or \quad \partial_s \eta = \Omega_{\mathbf{b}} - \eta \Omega_{\mathbf{nb}} + \eta^2 \Omega_{\mathbf{n}} \quad with \quad \mu = -1/\eta) \quad (26)$$

Thus, in the moving coordinates, the Jacobi identity or, equivalently, the existence of Poisson structure is expressible as a differential equation in arclength coordinates only. It may be interesting to note that the equation named after Jacopo Francesco Riccati originated from his investigations of curves whose radii of curvature depend only on a single variable [27]. The disappearance of the moving coordinates n and b from the Jacobi identity will become clear in the last section. In fact, they correspond to local conserved quantities and, as discussed in [11] for the globally integrable cases, may appear arbitrarily in the Poisson vectors.

The Riccati equation (25) is equivalent to the linear second order equation

$$\partial_{ss}^2 u - \left(\frac{\partial_s \Omega_{\mathbf{b}}}{\Omega_{\mathbf{b}}} + \Omega_{\mathbf{nb}} \right) \partial_s u + \Omega_{\mathbf{n}} \Omega_{\mathbf{b}} u = 0 \quad \text{if } \Omega_{\mathbf{b}} \neq 0 \quad (27)$$

$$\text{(or } \partial_{ss}^2 v - \left(\frac{\partial_s \Omega_{\mathbf{n}}}{\Omega_{\mathbf{n}}} - \Omega_{\mathbf{nb}} \right) \partial_s v + \Omega_{\mathbf{n}} \Omega_{\mathbf{b}} v = 0 \quad \text{if } \Omega_{\mathbf{n}} \neq 0 \text{)} \quad (28)$$

with the solutions being related by

$$\mu = -\frac{\partial_s \ln u}{\Omega_{\mathbf{b}}} \quad \text{if } \Omega_{\mathbf{b}} \neq 0 \quad \text{(or } \eta = -\frac{\partial_s \ln v}{\Omega_{\mathbf{n}}} \quad \text{if } \Omega_{\mathbf{n}} \neq 0 \text{).} \quad (29)$$

For the Poisson vectors of the above proposition at least one of the equations in (27) possesses two linearly independent solutions.

The emergence of the Riccati equation as the Jacobi identity may be interpreted as a relation between nonlinearity and superposition. The Jacobi identity as well as the compatibility condition for Poisson vectors are nonlinear restrictions on some linear combinations of basis vectors. Both are scalar equations in three dimensions. The Riccati equation, on the other hand, is known to be the only scalar equation admitting a nonlinear superposition principle [28].

4 Compatibility conditions

To construct the bi-Hamiltonian structure, the Poisson vectors constructed from the linearly independent solutions of the Riccati equation must be compatible. That means, their linear combinations must also satisfy the Jacobi identity. Although, the multiplicative factors are left arbitrary in the construction of Poisson vectors they become important in the compatibility condition. Apart from the general result discussed in section two, we shall restrict ourselves to the case where c is a constant in the combination $\mathbf{J}_1 + c\mathbf{J}_2$. We shall show that the compatibility follows from the Hamilton's equations.

First, we have the following result obtained by direct computation from the compatibility condition

Proposition 2 *Let α_i and μ_i be non zero and be different functions for $i = 1, 2$. For $\Omega_{\mathbf{b}} \neq 0$, the Poisson vectors $\mathbf{J}_i = \alpha_i(\mathbf{n} + \mu_i \mathbf{b})$ are compatible if*

$$\partial_s \ln \frac{\alpha_2}{\alpha_1} = (\mu_1 - \mu_2) \Omega_{\mathbf{b}}. \quad (30)$$

In obtaining Eq 30 we used the Riccati equations to eliminate derivatives of functions μ_1 and μ_2 . Next result shows that above equation is always satisfied, via Hamilton's equations, by Poisson vectors constructed from the solutions of the Riccati equation.

Proposition 3 *Let $\mathbf{J} = \alpha(\mathbf{n} + \mu\mathbf{b})$ and H define a Poisson structure for the dynamical system associated with \mathbf{v} . Then*

$$\partial_s \ln \frac{\|\mathbf{v}\|}{\alpha} - \mathbf{n} \cdot \nabla \times \mathbf{b} = \mu \Omega_{\mathbf{b}} \quad (31)$$

Proof. *With the Poisson vector in the assumptions we write the dynamical system as $\mathbf{t} = \|\mathbf{v}\|^{-1} \mathbf{J} \times \nabla H = \|\mathbf{v}\|^{-1} \alpha(\mathbf{n} + \mu\mathbf{b}) \times \nabla H$. Taking cross products with \mathbf{n} and \mathbf{b} we get*

$$\mathbf{b} = \|\mathbf{v}\|^{-1} (\alpha \nabla H - \mathbf{J}(\mathbf{n} \cdot \nabla H)) \quad \mathbf{n} = -\|\mathbf{v}\|^{-1} (\alpha \mu \nabla H - \mathbf{J}(\mathbf{b} \cdot \nabla H)) \quad (32)$$

from which we obtain $\nabla H = \alpha^{-2}(1 + \mu^2)^{-1}(\mathbf{J}(\mathbf{J} \cdot \nabla H) + \mathbf{J}^\perp \cdot \|\mathbf{v}\|)$. Here, we define $\mathbf{J}^\perp = \alpha(\mathbf{b} - \mu\mathbf{n})$ for convenience. The integrability condition $\nabla \times \nabla H = 0$ for the Hamiltonian function results, after taking dot product with \mathbf{J} , in

$$\partial_s \ln \frac{\|\mathbf{v}\|}{\alpha^2(1 + \mu^2)} = \frac{\mathbf{J} \cdot \nabla \times \mathbf{J}^\perp}{\alpha^2(1 + \mu^2)^2} \quad (33)$$

where we used $\mathbf{J} \cdot \mathbf{J}^\perp = 0$. The manipulations leading to the result is now straightforward and requires only the use of Jacobi identity for the derivative of μ . ■

The proof of compatibility becomes obvious once we write equation 31 for each Poisson vector and subtract them. Similar results may be obtained for the Poisson vectors in the form $\mathbf{J} = \beta(\mathbf{b} - \eta\mathbf{n})$ by assuming $\Omega_{\mathbf{n}} \neq 0$ and using the Riccati equation for the variable η .

We shall analyse the case $\Omega_{\mathbf{b}} = 0$. This is the condition for the unit vector \mathbf{b} to satisfy the Jacobi identity. On the other hand, $\Omega_{\mathbf{b}} = 0$ reduces the Riccati equation to a linear first order equation resulting in one linearly independent solution for the construction of a Poisson vector of the form $\mathbf{J} = \alpha(\mathbf{n} + \mu\mathbf{b})$.

Proposition 4 *The compatibility condition for Poisson vectors $\mathbf{J} = \alpha(\mathbf{n} + \mu\mathbf{b})$ and \mathbf{b} is $\partial_s \alpha + \Omega_{\mathbf{n}\mathbf{b}} = 0$. This is satisfied by the Hamilton's equations.*

Proof. *The equation can easily be obtained from the compatibility condition. Equation 31 with $\Omega_{\mathbf{b}} = 0$ holds for the Hamiltonian structure for \mathbf{J} . For the Hamiltonian structure with the Poisson vector \mathbf{b} , we have $\mathbf{t} = \|\mathbf{v}\|^{-1} \mathbf{b} \times \nabla H$ for some Hamiltonian function H . Taking cross-product with \mathbf{b} , solving for ∇H and taking the dot product of the equation resulting from the integrability condition $\nabla \times \nabla H = 0$, we find $\partial_s \ln \|\mathbf{v}\| = -\mathbf{b} \cdot \nabla \times \mathbf{n}$ which yields the result. ■*

Thus, the Poisson vectors obtained from solutions of Riccati equations are always compatible.

5 Bi-Hamiltonian structure

We shall combine the results of the previous two sections on construction of Poisson vectors and their compatibility to present the main result. This will include the remaining case where the Poisson vectors are defined by the normal and bi-normal unit vectors of the Serret-Frenet triad. In connection with this particular structure we shall first relate the present work to the existing examples of bi-Hamiltonian dynamical systems in the literature and then present the local form of the Nambu mechanics.

Proposition 5 *Every three dimensional dynamical system possesses two compatible Poisson vectors.*

Proof. *If both $\Omega_{\mathbf{n}}$ and $\Omega_{\mathbf{b}}$ are non-zero, then any of the two Riccati equations which produce the same result by construction, give two Poisson structures coming from the linearly independent solutions of the corresponding second order equation. If $\Omega_{\mathbf{b}} = 0$ (or $\Omega_{\mathbf{n}} = 0$) then the first (the second) Riccati equation becomes linear with one linearly independent solution. Note that the sum $\Omega_{\mathbf{nb}}$ of the cross-helicities of the normal vectors involves as an integrating factor in the integration for this Poisson structure. The other Poisson structure is defined by the bi-normal vector \mathbf{b} (or the normal vector \mathbf{n}) since $\Omega_{\mathbf{b}} = 0$ (or $\Omega_{\mathbf{n}} = 0$) is the Jacobi identity for this. The compatibility conditions for these cases are shown to be satisfied via Hamilton's equations in the previous section. If, we have both $\Omega_{\mathbf{n}}=0$ and $\Omega_{\mathbf{b}} = 0$ then \mathbf{n} and \mathbf{b} satisfies Jacobi identity and they become Poisson vectors we sought. The compatibility condition is $\Omega_{\mathbf{nb}} = 0$. This implies that the function μ (or η) must be a non-zero constant in the vector \mathbf{J} for which the Riccati equation is now trivial. The Hamilton's equations with the Poisson vectors \mathbf{n} and \mathbf{b} implies $\partial_s \ln \|\mathbf{v}\| = \mathbf{n} \cdot \nabla \times \mathbf{b}$ and $\partial_s \ln \|\mathbf{v}\| = -\mathbf{b} \cdot \nabla \times \mathbf{n}$, respectively. Eliminating the vector \mathbf{v} , we obtain the compatibility condition $\Omega_{\mathbf{nb}} = 0$.*

■

The last case of the theorem gives clues to relate the present work to the existing bi-Hamiltonian dynamical systems in the literature, in particular, to the Nambu mechanics. Recall that the three dimensional dynamical systems admitting bi-Hamiltonian structure with (\mathbf{J}_1, H_2) and (\mathbf{J}_2, H_1) are of the form of equation 11 with \mathbf{J}_1 and \mathbf{J}_2 satisfying Frobenius integrability conditions as Jacobi identities. In case that these vectors are globally integrable they are related to the conserved Hamiltonians by $\mathbf{J}_i = \varphi_i \nabla H_i$ $i = 1, 2$ for some arbitrary non-zero functions φ_i . In this case, the dynamical system has the form $\mathbf{v} = \psi \nabla H_1 \times \nabla H_2$ first studied by Nambu in [2]. All explicitly constructed bi-Hamiltonian systems in three dimension has this form [6], [7], [11], [30]. The flow lines coincide with the intersections of the surfaces defined by constant values of the Hamiltonians. Any constant appearing in this bi-Hamiltonian picture can be taken as arbitrary functions of H_1 and H_2 .

Our aim is to find the local version of this bi-Hamiltonian representation in three dimensions. This we shall do by first proving the more general result that

the local existence of a Hamiltonian function of the proposed form is equivalent to the existence of a Poisson vector (see also [29]).

Proposition 6 *There exists non-zero functions H and φ with $\mathbf{J} = \varphi \nabla H$ whenever $\mathbf{J} = \mathbf{n} + \mu \mathbf{b}$ satisfies the Jacobi identity.*

Proof. *The condition $\nabla \times \nabla H \equiv 0$ together with the form of \mathbf{J} gives $\nabla \times (\mathbf{n} + \mu \mathbf{b}) = \mathbf{a} \times (\mathbf{n} + \mu \mathbf{b})$ where $\mathbf{a} = \nabla \ln \sqrt{1 + \mu^2} / \varphi$. Taking dot products with the unit vectors of the Serret-Frenet triad we obtain three equations one of which is algebraic and the other two expressing the s and n derivatives of the function μ . All three contains terms involving the vector \mathbf{a} . Eliminating this term for the s derivative results in the Riccati equation 25 for the Poisson vector \mathbf{J} . The remaining two equations determine the function φ and the dependence of μ on the variable n . ■*

Thus, finding two linearly independent solutions of the Riccati equation completely determines the local bi-Hamiltonian structure.

In the local picture of the present work the normal coordinates n and b represents the local conserved quantities and they appear arbitrarily in the Poisson structures. The normal vectors \mathbf{n} and \mathbf{b} defining the bi-Hamiltonian structure in the last case thus corresponds, in the globally integrable case, to the gradients of Hamiltonian functions defining Poisson vectors. Therefore, the manifestly bi-Hamiltonian equation $\mathbf{t} = \mathbf{n} \times \mathbf{b}$ corresponding to the last case of the above proposition is the local version of the Nambu structure.

References

- [1] P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer-Verlag, New York, 1993.
- [2] Y. Nambu, Generalized Hamiltonian Dynamics, Phys. Rev. D 7 (1973) 2405-2412
- [3] M. Kus, Integrals of motion for the Lorenz system, J. Phys. A: Math. Gen. 16 (1983) L689-L691.
- [4] J. M. Strelcyn and S. Wojciechowski, A method of finding integrals for three-dimensional dynamical systems, Phys. Lett. A133 (1988) 207-212.
- [5] B. Grammaticos, J. Moulin-Ollagnier, A. Ramani, J. M. Strelcyn and S. Wojciechowski, Integrals of quadratic ordinary differential equations in \mathbb{R}^3 : The Lotka-Volterra system, Physica A 163 (1990) 683-722.
- [6] Y. Nutku, Hamiltonian structure of the Lotka-Volterra equations, Phys. Lett. A, 145 (1990) 27-28.

- [7] Y. Nutku, Bi-Hamiltonian structure of the Kermack-McKendrick model for epidemics, *J. Phys. A: Math. Gen.* 23 (1990) L1145-L1146.
- [8] H. J. Giacomini, C. E. Repetto and O. P. Zandron, Integrals of motion for three-dimensional non-Hamiltonian dynamical systems, *J. Phys. A: Math. Gen.* 24 (1991) 4567-4574.
- [9] S. A. Hojman, Quantum algebras in classical mechanics, *J. Phys. A: Math. Gen.*, 24 (1991) L249-L254
- [10] H. Gümral, PhD Thesis, Bilkent University, Ankara, 1991.
- [11] H. Gümral and Y. Nutku, Poisson structure of dynamical systems with three degrees of freedom, *J. Math. Phys.* 34 (1993) 5691-5723.
- [12] B. Hernández-Bermejo and V. Fairén, Separation of variables in the Jacobi identities, *Phys. Lett. A* 271 (2000) 258-263.
- [13] B. Hernández-Bermejo, New solutions of the Jacobi equations for three-dimensional Poisson structures, *J. Math. Phys.* 42 (2001) 4984-4996.
- [14] B. Hernández-Bermejo, Characterization and global analysis of a family of Poisson structures, *Phys. Lett. A* 355 (2006) 98-103
- [15] B. Hernández-Bermejo, New solution family of the Jacobi equations: Characterization, invariants, and global Darboux analysis, *J. Math. Phys.* 48 (2007) 022903-022914.
- [16] F. Haas and J. Goedert, On the Generalized Hamiltonian Structure of 3D Dynamical Systems, *Phys. Lett. A* 199 (1995) 173-179.
- [17] A. Weinstein, The local structure of Poisson manifolds, *J. Diff. Geom.* 18 (1983) 523-557.
- [18] G. R. W. Quispel and H. W. Capel, Solving ODEs numerically while preserving a first integral, *Phys. Lett. A* 218 (1996) 223-228.
- [19] B. Karasözen, Poisson integrators, *Mathematical and Computer Modelling*, 40 (2004) 1225-1244.
- [20] P. Tondeur, *Foliations on Riemannian Manifolds*, Springer, Berlin, 1988.
- [21] B. L. Reinhart, *Differential Geometry of Foliations*, Springer, Berlin, 1983.
- [22] H. E. Moses, Eigenfunctions of the Curl Operator, Rotationally Invariant Helmholtz Theorem, and Applications to Electromagnetic Theory and Fluid Mechanics, *SIAM J. Appl. Math.* 21 (1971) 114-144.
- [23] I. M. Benn and J. Kress, Force-free fields from Hertz potentials, *J. Phys. A: Math. Gen.* 29 (1996) 6295-6304.

- [24] L.C. Garcia de Andrade, Topology of magnetic helicity in torsioned filaments in Hall plasmas, astro-ph/07104594
- [25] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge University Press, 1967.
- [26] V. I. Arnold and B. A. Khesin, Topological Methods in Hydrodynamics, Springer-Verlag, 1998.
- [27] E. L. Ince, Ordinary Differential Equations, Dover, New York, 1956.
- [28] T. Bountis, H. Segur and F. Vivaldi, Integrable Hamiltonian systems and the Painleve property, Phys. Rev. A25 (1982) 1257-1264.
- [29] B. Hernández-Bermejo, A constant of motion in 3D implies a local generalized Hamiltonian structure, Phys. Lett. A234 (1997) 35-40.
- [30] A. Ay, M. Gürses and K. Zheltukhin, Hamiltonian Equations in \mathbb{R}^3 , J. Math. Phys. 44 (2003) 5688-5705.